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Interacting Quantum and Classical Continuous Systems II. Asymptotic Behavior of the Quantum Subsystem

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In the framework of event-enhanced quantum theory the dynamical equation for the reduced density matrix of a quantum system interacting with a continuous classical system is derived. The asymptotic behavior of the corresponding dynamical semigroup is discussed. The example of a quantum–classical coupling on Lobatchevski space is presented.

KEY WORDS: Completely positive coupling; dissipative dynamics; long-time behavior .

1. INTRODUCTION

The interest in quantum dynamical semigroup is based basically on their two important properties: they can be applied for the description of irreversible processes and provide a convenient mathematical framework for the study of the approach to equilibrium of open quantum systems. In the second problem one tries to answer when the system, irrespectively of its initial state, evolves into one specific state. More precisely one tries to establish the existence of the limit of the expectation value $\lim_{t \to \infty} Tr\rho(t) A$ for all observables A, and to show the independence of these limits of the initial state ρ_0. Conditions for a dynamical semigroup to evolve into a unique stationary state were found in the case of N-level quantum systems[33, 34] or when a dynamical semigroup possesses a faithful normal state.[13, 14, 36] These results were next generalized by Frigerio and Verri[15]

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to the case when the recurrent subspace projection R asymptotically approaches the identity operator, and by Kümmerer, Nagel and Groh[16, 22] for dynamical semigroups possessing a faithful family of normal subinvariant states. To our knowledge the case of a dynamical semigroup having no normal stationary state (apart from an example given by Evans[12]) was not investigated.

It is our objective to examine such a case, which arises naturally in the framework of event enhanced quantum theory.[3–5] In this approach one starts with an explicit split between a quantum system, which is not directly observable, and a classical system where events happen. From the kinematical point of view the total system is described by the tensor product of a non-commutative quantum algebra of observables with a commutative algebra of functions defined on a classical phase space. A dynamical semigroup of completely positive operators on the total algebra replaces Schrödinger unitary dynamics. It ensures the flow of information from the quantum system to classical one and, on the other hand, the influence of classical variables on the evolution of the quantum system. The construction of the coupling operator in the case when the quantum system is characterized by generalized coherent states on a homogeneous space (and more generally: by a semispectral measure) was presented in ref. 28. In ref. 6 it was shown that the evolution of ensembles is described by a Markov–Feller process, the so-called piecewise deterministic process. The modification of the classical trajectories of motion was also discussed. In the present paper we concentrate on the asymptotic behavior of the reduced density matrices obtained from the total density operators by tracing over the classical variables. In Section 2 we derive the evolution equation for the reduced density matrix and show that it fits into the framework of the quantum stochastic processes of Davies.[7–9] Although these processes were introduced in order to describe rigorously certain measurement procedures, they also arise in the description of quantum systems interacting with classical systems. In Section 3 we determine the spectrum of the evolution generator and describe the asymptotic behavior of the dynamical semigroup. Concluding remarks are presented in Section 4.

2. THE DYNAMICAL EQUATIONS

2.1. The Framework

Let us briefly describe the framework for the classical-quantum coupling. At first we consider a classical system C with a finite number of degrees of freedom. Its phase space is a symplectic manifold (M, ω). The C^*-algebra $C_0(M)$ of continuous functions vanishing at infinity represents the complex observables of the system. Because it will be more convenient to consider von Neumann algebras we pass to the representation in the Hilbert space $\mathscr{H}_c = L^2(M, \mathscr{B}, \mu)$, where $\mathscr{B}$ is the Borel σ-algebra and $d\mu$ is the unique Borel measure determined by the volume form ω^n, $n = \dim M/2$. We assume that the classical algebra $\mathscr{A}_c$ equals to $C_0(M)'' = L^\infty(M, \mathscr{B}, \mu)$. Statistical states of C are then normed and positive elements of $L^1(M, \mathscr{B}, \mu)$. The time evolution of C is described by a flow on M, i.e., a mapping g: $(t, x) \to g_t(x)$ such that:

(a) $g: \mathbf{R} \times M \to M$ is smooth,

(b) for any t, g_t is a diffeomorphism of M,

(c) $t \to g_t$ is a group homomorphism.

Its generator is a complete vector field X on M. It gives an ultraweakly continuous one parameter group of automorphisms of $\mathscr{A}_c$: $f(x) \to f(g_t^{-1} x)$, $x \in M$. Its generator we denote by δ_c.

Now we come to the quantum system. Let us consider a quantum particle on a homogeneous configuration space $Q = G/K$, where G is a Lie group and K is a closed subgroup. We assume moreover that G and K are both unimodular. The quantum theory of such a system can be introduced by using the concept of generalized coherent states[29] (see also ref. 20). Let $(\pi, \mathscr{H}_q)$ be a unitary, strongly continuous and irreducible representation of G, such that for every $k \in K$ $\pi(k)\, \psi_0 = e^{ia(k)} \psi_0$ for some unit vector $\psi_0 \in \mathscr{H}_q$. It follows that for each $q \in Q$ we have a one-dimensional projector $P_q = |\pi(g)\, \psi_0\rangle \langle \psi_0 \pi(g)|$, where $[g] = q$. We assume that the system of coherent states is square integrable and normalized, i.e.

$$\int_Q d\alpha(q)\, P_q = \mathbf{1}$$

in the strong sense, where $d\alpha$ is a unique G-invariant and σ-finite Borel measure on Q. We also assume that for every $q \in Q$ the reproducing kernel $q' \to K(q, q') = \langle q, q' \rangle$ vanishes only on a set of α-measure zero. This assumption can be often easily verified. For example in the case of unitary group $U(n)$ with the natural representation in $\mathbf{C}^n$ there is

$$Q = U(n)/U(n-1) \times U(1) = \mathbf{C}P^{n-1}$$

and so $|\psi'\rangle \to \langle \psi, \psi' \rangle = 0$ only when $|\psi'\rangle \in \mathbf{C}P^{n-2}$, which is imbeded into Q. And $\alpha(\mathbf{C}P^{n-2}) = 0$. In many infinite dimensional examples the space Q is a Kähler manifold and the orbit $|\pi(g)\, \psi_0\rangle$ is a complex submanifold of $\mathbf{C}P(\mathscr{H}_q)$. Such a representation is called a Kähler coherent state representation.[23] In this case the representation $(\pi, \mathscr{H}_q)$ has a holomorphic realization and so the reproducing kernel is a holomorphic function. Hence the required assumption is clearly satisfied. The quantum algebra $\mathscr{A}_q$ is defined as

$$\mathscr{A}_q = \left\{ \int f(q)\, P_q\, d\alpha(q),\, f \in C_c(Q) \right\}'' = \{P_q,\, q \in Q\}''$$

Proposition 2.1. $\mathscr{A}_q$ is equal to $B(\mathscr{H}_q)$, the algebra of all bounded and linear operators on $\mathscr{H}_q$.

Proof. It is enough to show that $\mathscr{A}_q$ contains all one-dimensional projectors. Because the representation π is irreducible so finite linear combinations of elements $\pi(g_i)\,\psi_0$ are dense in $\mathscr{H}_q$. Because $P_q P_{q'} = K(q, q')\, |q\rangle\langle q'|$ and $P_q P_{q'} \in \mathscr{A}_q$, so any $P = |\psi\rangle\langle\psi|$ is a weak limit of one-dimensional projectors belonging to $\mathscr{A}_q$. ∎

Statistical states of the quantum system are given by non-negative density matrices $\rho \in \mathscr{A}_q$ with $Tr(\rho) = 1$. The time evolution is given by $A \to e^{itH} A e^{-itH}$, where H is the operator closure of $(d\pi(h), D_G)$, $h \in \mathscr{G}$—the Lie algebra of group G, and D_G is the Gårding domain. Clearly H is a self-adjoint operator. The generator $i[H, \cdot]$ describing the evolution of the quantum system will be denoted by δ_q.

Let us now consider the joint system. For the total algebra $\mathscr{A}_T$ we take the tensor product $\mathscr{A}_T = \mathscr{A}_c \otimes \mathscr{A}_q$ as von Neumann algebras on $\tilde{\mathscr{H}} = \mathscr{H}_c \otimes \mathscr{H}_q$. The set of states is equal to

$$\mathscr{S}_T = \left\{ \tilde{\rho} \in \mathscr{A}_{T*} : \tilde{\rho}(x) \in Tr(\mathscr{H}_q)_+ \text{ a.e. and } \int_M Tr(\tilde{\rho}(x))\, d\mu(x) = 1 \right\}$$

where $Tr(\mathscr{H}_q)$ denotes the Banach space of trace class operators on $\mathscr{H}_q$. The mean value of $\tilde{A} \in \mathscr{A}_T$ in a state $\tilde{\rho} \in \mathscr{S}_T$ is given by

$$\langle \tilde{A} \rangle_{\tilde{\rho}} = \int_M d\mu(x)\, Tr[\tilde{A}(x)\, \tilde{\rho}(x)]$$

Now let us discuss the time evolution of the total system. The total generator consists of three parts: $\delta_c \otimes id$, $id \otimes \delta_q$ and a coupling operator L, which describes the interaction between the classical and the quantum system. To construct L we assume that to every point $q \in Q$ corresponds a shift on the phase space M. By shift we mean a morphism of $(M, \mathscr{B}, \mu)$, i.e., a bijective map $h_q : M \to M$ such that h_q and h_q^{-1} are measurable and leave the measure $d\mu$ invariant. Moreover we require that for any $f \in C_c(M)$ and any $x \in M$ the mapping $q \to f(h_q^{-1} x)$ is measurable.

Proposition 2.2. Suppose $\tilde{A} \in \mathscr{A}_T$. Let

$$L(\tilde{A})(x) = \lambda \int_Q d\alpha(q)\, P_q \tilde{A}(h_q x)\, P_q - \lambda \tilde{A}(x)$$

where $\lambda > 0$ is the coupling constant. Then L is a bounded and complete dissipation such that $L(\tilde{\mathbf{1}}) = 0$, where $\tilde{\mathbf{1}}$ is the unit in $\mathscr{A}_T$. Moreover L is normal.

Proof. It follows from a more general construction presented in ref. 28, if we put the measure $\nu_x = \lambda^{1/2} \delta_e$, where δ_e is the Dirac measure concentrated on the neutral element in G. ∎

Corollary. The operator

$$B = \delta_c \otimes id + id \otimes \delta_q + L$$

generates a dynamical semigroup T_t on the algebra $\mathscr{A}_T$.

By a dynamical semigroup we understand a weak*-continuous semigroup of contractive, completely positive and normal operators.

Because T_t is normal it admits a preadjoint operator on the predual space $\mathscr{A}_{T*}$. We denote it by T_{t*}. The semigroup $t \to T_{t*}$ is strongly continuous and its generator is given by $B_* = \delta_c^{ad} \otimes id + id \otimes \delta_q^{ad} + L_*$, where

$$L_* \tilde{\rho}(x) = \lambda \int_Q d\alpha(q)\, P_q \tilde{\rho}(h_q^{-1} x)\, P_q - \lambda \tilde{\rho}(x)$$

2.2. Tracing over Classical Variables

Now we derive the evolution equation for the reduced density matrix $\rho_t = \int_M d\mu(x)\, \tilde{\rho}_t(x)$ corresponding to the quantum system.

$$\dot{\rho}_t = \int_M d\mu(x)\, \dot{\tilde{\rho}}_t(x) = \int_M d\mu(x) [\delta_c^{ad} \otimes id + id \otimes \delta_q^{ad} + L_*]\, \tilde{\rho}_t(x)$$

Let us notice that for simple tensors $\phi \otimes \rho$, $\phi \in L^1(M, \mathscr{B}, \mu)$, $\rho \in Tr(\mathscr{H}_q)$ such that $\phi \in D(\delta_c^{ad})$ and $\rho \in D(\delta_q^{ad})$ there is

$$\int_M d\mu(x) (\delta_c^{ad} \otimes id)(\phi \otimes \rho)(x) = \rho \int_M d\mu(x)\, \phi(x)\, \delta_c(\mathbf{1})(x) = 0$$

and

$$\int_M d\mu(x)(id \otimes \delta_q^{ad})(\phi \otimes \rho)(x) = \delta_q^{ad}\left(\rho \int_M d\mu(x)\, \phi(x)\right)$$

Because $\tilde{\rho}_t \in D(B_*)$ and B_* is the closure of the corresponding operator defined on the algebraic tensor product $D(\delta_c^{ad}) \otimes D(\delta_q^{ad})$ we have that

$$\tilde{\rho}_t = \lim_{n \to \infty} \sum_{i=1}^{n} \phi_i \otimes \rho_i$$

and

$$\lim_{n \to \infty} (\delta_c^{ad} \otimes id + id \otimes \delta_q^{ad}) \left(\sum_{i=1}^{n} \phi_i \otimes \rho_i \right) = (\delta_c^{ad} \otimes id + id \otimes \delta_q^{ad})\, \tilde{\rho}_t$$

Because the tracing operator is continuous and δ_q^{ad} is closed we conclude that

$$\int_M d\mu(x)(\delta_c^{ad} \otimes id + id \otimes \delta_q^{ad})\, \tilde{\rho}_t(x) = \delta_q^{ad}\left(\int_M d\mu(x)\, \tilde{\rho}_t(x)\right)$$

Finally

$$\int_M d\mu(x)(L_* \tilde{\rho}_t)(x) = \lambda \int_M d\mu(x) \int_Q d\alpha(q)\, P_q \tilde{\rho}_t(h_q^{-1} x)\, P_q - \lambda \rho_t$$

$$= \lambda \int_Q d\alpha(q)\, P_q \rho_t P_q - \lambda \rho_t$$

Hence the time evolution equation for ρ_t is given by

$$\dot{\rho}_t = -i[H, \rho_t] + \lambda \int_Q d\alpha(q)\, P_q \rho_t P_q - \lambda \rho_t$$

For simplicity we denote the above generator also by B_* and its dissipative part by L_*. Hence $B_* = -i[H, \cdot] + L_*$. The semigroup it generates we denote also by T_{t*} and its adjoint acting on $B(\mathcal{H}_q)$ by T_t.

2.3. Connection with the Davies Quantum Processes

Let us consider a quantum stochastic process on $(Q, Tr(\mathcal{H}_q))$ as defined in ref. 7. By Theorem 4.7 from ref. 7 such a process is uniquely characterized by a generator Z of a strongly continuous one parameter semigroup on $\mathscr{H}_q$ and by a bounded stochastic kernel J. Let us recall that a bounded stochastic kernel is a bounded positive σ-additive measure on the σ-algebra of Borel sets in Q with values in bounded linear operators on $Tr(\mathscr{H}_q)$. The only condition they have to satisfy is

$$Tr[J(Q, P_\psi)] = -2\,\mathrm{Re}\langle Z\psi, \psi\rangle$$

for all normalized $\psi \in D(Z)$. Here $P_\psi = |\psi\rangle\langle\psi|$. At first let us notice that for any Borel subset $E \subset Q$ and $A \in B(\mathscr{H}_q)$ the following formula

$$Tr[J(E, \rho)\, A] = \int_E Tr(P_q \rho P_q A)\, d\alpha(q)$$

define a bounded stochastic kernel (Theorem 5.1 in ref. 7). Putting $Z = iH - \frac{1}{2}\mathbf{1}$ we obtain that

$$Tr[J(Q, P_\psi)] = \int_Q d\alpha(q)\, Tr(P_q P_\psi P_q) = 1 = -2\,\mathrm{Re}\langle Z\psi, \psi\rangle$$

so Z and J are infinitesimal generators of some quantum stochastic process. The strongly continuous semigroups on $Tr(\mathscr{H}_q)$ associated with the process and expressed in terms of Z and J reads

$$T_t^p(\rho) = e^{tZ^*} \rho e^{tZ} + tJ(Q, \rho) + o(t)$$

for small t. Its generator, let say B^p, is given by

$$B^p(\rho) = -i[H, \rho] - \rho + \int_Q d\alpha(q)\, P_q \rho P_q$$

and coincides with B_*. Finally, let us notice that from the identity $P_{gq} = \pi(g)\, P_q \pi(g)^*$ it follows that J is covariant with respect to the representation π, i.e., the equality

$$J(gE, \rho) = \pi(g)\, J(E, \pi(g)^* \rho \pi(g))\, \pi(g)^*$$

holds for all $\rho \in Tr(\mathscr{H}_q)$, $E \in \mathscr{B}(Q)$ and $g \in G$.[8]

3. THE ASYMPTOTIC BEHAVIOR OF T_{t*} AND T_t

3.1. Preliminaries

For a closed densely defined linear operator A with domain $D(A)$ in a Banach space X we denote by $\rho(A)$ its resolvent set. The resolvent

$R(z, A) = (z\mathbf{1} - A)^{-1}$ is a holomorphic function in z for all $z \in \rho(A)$. We call z_0 a pole of the resolvent if $R(z, A)$ has a Laurent expansion

$$R(z, A) = \sum_{k=-1}^{\infty} B_k (z - z_0)^k, \qquad B_{-1} \neq 0$$

for all z in some neighborhood of z_0. The operator B_{-1} is called the residue of $R(z, A)$ at z_0. The spectrum $\sigma(A) = \mathbf{C} \backslash \rho(A)$ will be divided into two parts:

the approximate point spectrum

$$A\sigma(A) = \{\lambda \in \mathbf{C} : \lambda - A \text{ is not injective or } (\lambda - A)\, D(A) \text{ is not closed in } X\}$$

and the residual spectrum

$$R\sigma(A) = \{\lambda \in \mathbf{C} : (\lambda - A)\, D(A) \text{ is not dense in } X\}$$

Clearly we have $\sigma(A) = A\sigma(A) \cup R\sigma(A)$ but the union need not be disjoint. Another partition of $\sigma(A)$, more used by physicists, is given by:

the point spectrum

$$\sigma_p(A) = \{\lambda \in \mathbf{C} : \lambda - A \text{ is not injective}\}$$

the continuous spectrum

$$\sigma_c(A) = \{\lambda \in \mathbf{C} : (\lambda - A)\, D(A) \text{ is dense but not closed in } X\}$$

and the (strict) residual spectrum

$$\sigma_r(A) = \{\lambda \in \mathbf{C} : \lambda - A \text{ is injective and } (\lambda - A)\, D(A) \text{ is not dense in } X\}$$

Now $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$ are mutually disjoint and their union is also equal to $\sigma(A)$. Between these sets exist the following relations: $\sigma_p \subset A\sigma$, $\sigma_r \subset R\sigma \subset \sigma_p \cup \sigma_r$ and $\sigma_c \subset A\sigma$. The usefulness of $A\sigma$ and $R\sigma$ follows from the following:

Proposition 3.1. (a) $\lambda \in A\sigma(A)$ iff there exists a sequence $\{x_n\}$, $x_n \in D(A)$, $\|x_n\| = 1$ such that $\lim_n \|Ax_n - \lambda x_n\| = 0$.

(b) the topological boundary $\partial\sigma(A)$ of $\sigma(A)$ is contained in $A\sigma(A)$.

(c) $R\sigma(A) = \sigma_p(A^*)$, where A^* is the adjoint operator in X^*. Moreover $\sigma(A) = \sigma(A^*)$.

Suppose that X is a complex Banach space such that $X = X_R + iX_R$, where X_R is a real Banach subspace of X. We say that (X_R, X_+) is an ordered Banach space if X_+ is a closed cone in X_R (a cone of positive elements). X_+ is called proper if $X_+ \cap -X_+ = \{0\}$, generating if $X_R = X_+ - X_+$ and weakly generating if the norm closure of $X_+ - X_+$ is equal to X_R. Let (X_R^*, X_+^*) denote the ordered dual Banach space. A point $x \in X_+$ is called a quasi-interior point if $\omega(x) > 0$ for all $\omega \in X_+^* \setminus \{0\}$. It follows that the set of quasi-interior points qu.int.X_+ is either empty or norm dense in X_+. A subcone C of X_+ is said to be hereditary if for any $y \in X_+$, $y \leqslant x$ and $x \in C$ implies that $y \in C$. For arbitrary subcones $C \subset X_+$ and $C^* \subset X_+^*$ we define

$$C^\perp = \{\omega \in X_+^* : \omega(x) = 0 \ \forall x \in C\}$$

$$(C^*)^\top = \{x \in X_+ : \omega(x) = 0 \ \forall \omega \in C^*\}$$

X_+ is said to be sharp if each hereditary subcone C of X_+ for which $C^\perp = \{0\}$ is norm dense in X_+. Similarly, we say that X_+^* is *-sharp if each hereditary subcone C^* of X_+^* for which $(C^*)^\top = \{0\}$ is weak* dense in X_+^*. It is known that $Tr(\mathscr{H}_q)$ is sharp and $B(\mathscr{H}_q)$ is *-sharp (Example 2.5.5 in ref. 2). Suppose that T_t, $t \geqslant 0$ is a strongly continuous semigroup of contractions in X. T_t is said to be strictly positive if $T_t(X_+ \setminus \{0\}) \subset$ qu.int.X_+ for all $t > 0$, norm-ergodic if for each $x \in X_+ \setminus \{0\}$ the smallest T-invariant hereditary subcone of X_+ containing x is norm dense in X_+, norm-irreducible if there is no proper norm closed T-invariant hereditary subcone of X_+. Similarly, we say that T_t^* is weak*-irreducible if there is no proper weak* closed T^*-invariant hereditary subcone of X_+^*.

Finally, let us notice that since T_t is a contraction, the complex half-plane $\{z \in \mathbf{C} : \operatorname{Re} z > 0\}$ is contained in $\rho(A)$, where A is the generator of T_t. The set $\sigma(A) \cap i\mathbf{R}$ we call the peripheral spectrum of A. It plays an important role in the analysis of the asymptotic behavior of the semigroup T_t. At last, by Fix(T) we denote the set $\{x \in X : T_t(x) = x\}$ for all $t \geqslant 0$.

3.2. The Peripheral Spectrum of L_* and L

Because the Hamiltonian $H = \overline{d\pi(h)}$ and we have not specified $h \in \mathscr{G}$, so we intend to control the behavior of the semigroup T_{t*} and its adjoint T_t by the properties of the semigroup $S_{t*} = \exp(tL_*)$. Let us notice that $(S_{t*})^* = S_t = \exp(tL)$.

Lemma 3.2. If $\rho \in Tr(\mathscr{H}_q)_+ \setminus \{0\}$ then $S_{t*}(\rho)$ is faithful for any $t > 0$.

Proof. It is enough to show that $S_{t*}(P)$ is faithful for each one-dimensional projector P. Because L_* is bounded so

$$S_{t*}(P) = P + tL_*(P) + \frac{t^2}{2!} L_*^2(P) + \cdots$$

$$= e^{-t} \left[P + t \int_Q d\alpha(q)\, f_1(q)\, P_q + \frac{t^2}{2!} \int_Q d\alpha(q)\, f_2(q)\, P_q + \cdots \right]$$

where $f_1(q) = Tr(PP_q)$, $f_2(q) = \int d\alpha(q')\, Tr(P_q P_{q'})\, f_1(q')$ and so on. Suppose that there exists a normalized vector $\psi \in \mathcal{H}_q$ such that $S_{t*}(P)\,\psi = 0$. Then

$$\langle \psi, S_{t*}(P)\,\psi \rangle = e^{-t} \left[Tr(PP_\psi) + t \int_Q d\alpha(q)\, f_1(q)\, Tr(P_q P_\psi) + \cdots \right] = 0$$

for $P_\psi = |\psi\rangle\langle\psi|$. Because each summand is nonnegative so they all have to be zero. Let us consider the third one.

$$\int_Q d\alpha(q)\, f_2(q)\, Tr(P_q P_\psi) = 0 \tag{1}$$

At first we show that $f_2(q) > 0$ for all q. Suppose that there exists q_0 such that $f_2(q_0) = 0$. It means that

$$\int_Q d\alpha(q)\, Tr(P_{q_0} P_q)\, f_1(q) = 0$$

Because $q \to |\langle q_0, q \rangle|^2$ vanishes only on a set of α-measure zero so $f_1(q) = 0$ for almost every q. But $\int d\alpha(q)\, f_1(q) = 1$ so we get a contradiction. Hence $f_2(q) > 0$ and Eq. (1) is satisfied only if $Tr(P_q P_\psi) = 0$ for all q. But it is impossible since again $\int d\alpha(q)\, Tr(P_q P_\psi) = 1$. Hence $S_{t*}(P)$ is faithful. ∎

Corollary. S_{t*} is strictly positive.

Lemma 3.3. S_{t*} is norm-irreducible in $Tr(\mathcal{H}_q)$.

Proof. It is equivalent to show that S_t is weak*-irreducible. By Proposition 2.1 $\{P_q,\ q \in Q\}' = \mathbf{C}1$ so we may use Theorem 4.1 in ref. 12. Another argument is of the following type. Because the positive cone

$Tr(\mathscr{H}_q)_+$ is sharp and S_{t*} is strictly positive so the norm-irreducibility of S_{t*} follows from Theorem 2.5.1 in ref. 2. ∎

Remark. By the second argument we showed a little more, namely that S_{t*} is norm-ergodic.

Now we consider the peripheral spectrum of the generators L_* and L.

Lemma 3.4. $\sigma(L_*) \cap i\mathbf{R} \subset \{0\}$.

Proof. It is clear that $\sigma(L_*) \cap i\mathbf{R} \subset A\sigma(L_*)$. Suppose that there exists $\alpha \in \mathbf{R}$ such that $i\alpha \in A\sigma(L_*)$. Then there exists a sequence of $\phi_n \in Tr(\mathscr{H}_q)$ such that $\|\phi_n\|_{Tr} = 1$ and

$$L_*(\phi_n) = i\alpha\phi_n + h_n \tag{2}$$

with $\|h_n\|_{Tr} \to 0$. Every ϕ_n has the polar decomposition $\phi_n = U_n\, |\phi_n|$, where U_n is a partial isometry and $Tr\, |\phi_n| = 1$. Hence

$$\begin{aligned}\langle L_*(\phi_n), U_n^* \rangle &= \left\langle \int_Q d\alpha(q)\, P_q \phi_n P_q,\, U_n^* \right\rangle - 1 \\ &= i\alpha + \langle h_n, U_n^* \rangle\end{aligned}$$

taking the limit $n \to \infty$ we conclude that

$$\lim_{n\to\infty} \int_Q d\alpha(q)\, Tr(|\phi_n|\, P_q U_n^* P_q U_n) = 1 + i\alpha$$

and so

$$\lim_{n\to\infty} \left| \int_Q d\alpha(q)\, Tr(|\phi_n|\, P_q U_n^* P_q U_n) \right| = \sqrt{1+\alpha^2}$$

Every $|\phi_n|$ admits the representation

$$|\phi_n| = \sum_{k=1}^{\infty} a_k^{(n)} P_k^{(n)}, \qquad a_k^{(n)} \geqslant 0, \qquad \sum_{k=1}^{\infty} a_k^{(n)} = 1$$

where $P_k^{(n)} = |\psi_k^{(n)}\rangle\langle\psi_k^{(n)}|$ are one-dimensional orthogonal projectors. Hence

$$\left|\int_Q d\alpha(q)\, Tr(|\phi_n|\, P_q U_n^* P_q U_n)\right|$$
$$\leqslant \sum_{k=1}^{\infty} a_k^{(n)} \int_Q d\alpha(q)\, |Tr(P_k^{(n)} P_q U_n^* P_q U_n)|$$
$$= \sum_{k=1}^{\infty} a_k^{(n)} \int_Q d\alpha(q)\, |\langle q, \psi_k^{(n)}\rangle| \cdot |\langle q, U_n \psi_k^{(n)}\rangle| \cdot |\langle q, U_n^* q\rangle|$$
$$\leqslant \tfrac{1}{2} \sum_{k=1}^{\infty} a_k^{(n)} \int_Q d\alpha(q) (|\langle q, \psi_k^{(n)}\rangle|^2 + |\langle q, U_n \psi_k^{(n)}\rangle|^2)$$
$$= \tfrac{1}{2} \sum_{k=1}^{\infty} a_k^{(n)} \int_Q d\alpha(q) [\, Tr(P_q P_k^{(n)}) + Tr(P_q U_n P_k^{(n)} U_n^*)] = 1$$

Hence $\alpha = 0$. ▮

Lemma 3.5. If $\dim \mathscr{H}_q = \infty$ then $\mathrm{Fix}(S_*) = 0$.

Proof. Because $\{P_q,\ q \in Q\}' = \mathbf{C}\mathbf{1}$ so any $\rho \in Tr(\mathscr{H}_q)$ such that $L_*(\rho) = 0$ has to be proportional to the identity operator.[12] But $\mathscr{H}_q$ is infinite dimensional so $\rho = 0$. ▮

If $\dim \mathscr{H}_q = d$ is finite then there exists a unique invariant density matrix given by $\rho = (1/d)\,\mathbf{1}$. Since this case is well known we assume from now that $\dim \mathscr{H}_q = \infty$.

It is clear that $0 \in \sigma_p(L)$. Thus we obtained the following:

Theorem 3.6.

$$\sigma(L_*) \cap i\mathbf{R} = R\sigma(L_*) \cap i\mathbf{R} = A\sigma(L_*) \cap i\mathbf{R} = \{0\}$$
$$\sigma_p(L_*) \cap i\mathbf{R} = \sigma_c(L_*) \cap i\mathbf{R} = \varnothing$$
$$\sigma(L) \cap i\mathbf{R} = \sigma_p(L) \cap i\mathbf{R} = \{0\} \qquad \text{and} \qquad \mathrm{Fix}(S) = \ker(L) = \mathbf{C}\mathbf{1}$$

Proof. Only the last statement needs a proof but it follows from Theorem 3.1 in ref. 12. ▮

Remark. Let us notice that, in general, the peripheral spectrum of a semigroup generator has a nice structure. Its study was motivated by the Frobenius theorem. Let us recall his result: for an irreducible square matrix A such that its entries A_{ij} are nonnegative and with the spectral radius $r(A)$

the set of eigenvalues λ satisfying $|\lambda| = r(A)$ is cyclic.[32] A similar result was shown for a positive semigroup of contractions on a Banach lattice, hence on commutative C^*-algebras, (see Section C-III in ref. 25). Namely, it states that the peripheral spectrum of the generator is cyclic. Passing to the dynamical semigroups on noncommutative C^*-algebras the condition of the irreducibility has to be imposed again. If a weak*-irreducible and identity preserving semigroup e^{tA} with the preadjoint e^{tA_*} acts on a von Neumann algebra then assuming that e^{tA} is of Schwartz type ($e^{tA}(x^*x) \geqslant e^{tA}(x)^* e^{tA}(x)$) and that $\sigma_p(A_*) \cap i\mathbf{R} \neq \emptyset$ it was shown that $\sigma_p(A_*) \cap i\mathbf{R}$ is an additive subgroup of $i\mathbf{R}$ (Theorem 1.10 Section D-III in ref. 16). If we admit the case $\sigma_p(A_*) \cap i\mathbf{R} = \emptyset$, it can be shown that $\sigma_p(A) \cap i\mathbf{R}$ is an additive subgroup of $i\mathbf{R}$. Finally, we show that the semigroup T_{t*} has also the preadjoint semigroup. Let $K(\mathscr{H}_q)$ denote the space of all compact operators.

Proposition 3.7. $T_t\colon K(\mathscr{H}_q) \to K(\mathscr{H}_q)$.

Proof. Because $L|_{Tr} = L_*\colon Tr(\mathscr{H}_q) \to Tr(\mathscr{H}_q)$ and $Tr(\mathscr{H}_q)$ is dense in the operator norm in $K(\mathscr{H}_q)$, so the assertion follows by the continuity in the operator norm of L. ▌

Corollary. T_t is not norm-irreducible on $B(\mathscr{H}_q)$. The generator of the restricted semigroup $T_t|_K$ we denote by $L|_K$. Clearly we have the following.

Proposition 3.8. $\sigma(L|_K) \cap i\mathbf{R} = \sigma_c(L|_K) \cap i\mathbf{R} = \{0\}$.

3.3. The Ergodic Properties of T_{t*} and T_t

We will first establish the relationship between the semigroups T_t and S_t. Let $V_t = e^{itH}$. Because $H = \overline{d\pi(h)}$ so $V_t = \pi(g_t)$, where $g_t = \exp(th)$.

Proposition 3.9. $T_t(A) = V_t S_t(A)\, V_t^*$ for all $A \in B(\mathscr{H}_q)$.

Proof. Because the stochastic kernel J is π-invariant (Section 2.3), so $V_t L(A)\, V_t^* = L(V_t A V_t^*)$ and so $V_t S_{t'}(A)\, V_t^* = S_{t'}(V_t A V_t^*)$ for all t, t'. Hence

$$T_t(A) = e^{it[H,\cdot] + tL}(A) = V_t S_t(A)\, V_t^*$$

by the Trotter product formula. ▌

Now we are in position to prove the following:

Theorem 3.10:

$$\forall \rho \in Tr(\mathcal{H}_q) \quad \lim_{t \to \infty} \|T_{t*}\rho\|_{Tr} = |Tr\rho|$$

$$\forall A \in K(\mathcal{H}_q) \quad \lim_{t \to \infty} \|T_t A\|_{op} = 0$$

Proof. To prove the first statement we use the qualitative stability arguments (see Section 5.5 in ref. 27). Because $\sigma(L_*) \cap i\mathbf{R} = \{0\}$, so

$$\lim_{t \to \infty} \|S_{t*}\rho\|_{Tr} = \sup\{|Tr\rho A| : A \in \text{Fix}(S),\ \|A\|_{op} = 1\}$$

But $\text{Fix}(S) = \mathbf{C1}$, so $\lim_{t \to \infty} \|S_{t*}\rho\|_{Tr} = |Tr\rho|$. By Proposition 3.9 this also holds for the semigroup T_{t*}.

Since, by Proposition 3.8, $\sigma(L|_K) \cap i\mathbf{R} = \{0\}$ and $R\sigma(L|_K) \cap i\mathbf{R} = \emptyset$, the second statement follows from the ABLP stability theorem.[1, 24] ∎

Corollary. For any two densities matrices ρ_1 and ρ_2 we have that

$$\lim_{t \to \infty} \|T_{t*}(\rho_1) - T_{t*}(\rho_2)\|_{Tr} = 0$$

Passing now to the description of the ergodic properties of the semigroup T_{t*} let us notice that the set $\{T_{t*}\rho : t \geqslant 0\}$ is not relatively compact in the weak topology on $Tr(\mathcal{H}_q)$. To see this suppose that the contrary is true. By the same argument as in Lemma 3.5 we obtain that $\sigma_p(B_*) \cap i\mathbf{R} = \emptyset$. Hence the semigroup T_{t*} has 0 as a limit point for the weak operator topology.[26] So there is a sequence $\{t_n\}$ such that $\lim_n Tr(T_{tn*}\rho)\, A = 0$ for any $A \in B(\mathcal{H}_q)$. If we take $A = \mathbf{1}$ and ρ a density matrix we get a contradiction. It follows that the net $\{(1/t) \int_0^t T_{t*}\rho\, ds\}$, $t > 0$, has no limit points in the weak topology on $Tr(\mathcal{H}_q)$. This argument holds also in a more general case of a locally integrable semigroup (Theorem 7 in ref. 35).

Let T_t^* with generator B^* denote the adjoint semigroup acting on $B(\mathcal{H}_q)^*$. Because $\dim\ker(B) \leqslant \dim\ker(B^*)$, so there is at least one T^*-invariant functional in $B(\mathcal{H}_q)^*$.

Proposition 3.11. Every T^*-invariant functional $\omega \in B(\mathcal{H}_q)^*$ is singular. Let us recall that any $\omega \in B(\mathcal{H}_q)^*$ has a unique decomposition $\omega = \omega_n + \omega_s$ onto its normal and singular part.

Proof. By the assumption $T_t^*(\omega)=\omega$ for all $t\geqslant 0$. By Theorem 3.10 $\omega(A)=0$ for all $A\in K(\mathscr{H}_q)$ what ends the proof. ∎

This result implies that we have to consider the whole space $B(\mathscr{H}_q)^*$. Let χ_t, $t\geqslant 0$ be a net of probability measures on the Banach space $C_b(T)$ of continuous and uniformly bounded functions defined on the set $\{T_t, t\geqslant 0\}\subset\mathscr{L}(B(\mathscr{H}_q))$ and equipped with the sup norm. On $\mathscr{L}(B(\mathscr{H}_q))$ we put the topology of the pointwise convergence, so the multiplication $\mathscr{L}(B(\mathscr{H}_q))\times\mathscr{L}(B(\mathscr{H}_q))\to\mathscr{L}(B(\mathscr{H}_q))$ is separately continuous. The net χ_t is defined as follows

$$\chi_0(f)=f(\mathbf{1}) \qquad \text{and} \qquad \chi_t(f)=\frac{1}{t}\int_0^t ds\, f(T_s)$$

for all $f\in C_b(T)$. Let μ be a limit point of the net χ_t in the weak* topology on $C_b(T)^*$. Clearly, μ is an invariant mean on $C_b(T)$. Let j: $Tr(\mathscr{H}_q)\to B(\mathscr{H}_q)^*$ be the canonical imbeding. For any $\rho\in Tr(\mathscr{H}_q)$ we define a map

$$F_\rho: B(\mathscr{H}_q)\to C_b(T) \qquad F_\rho(A)(T_t)=\langle T_tA, j(\rho)\rangle$$

It is clear that F_ρ is linear and bounded. Let F_ρ^* denote its adjoint. Now we define an operator P: $Tr(\mathscr{H}_q)\to B(\mathscr{H}_q)^*$, $P(\rho)=F_\rho^*(\mu)$. It is a bounded operator such that $P=PT_{t*}=T_t^*P$ for all $t\geqslant 0$. Moreover $P(\rho)$ belongs to the closure in the weak* topology of the convex hull of $\{T_t^*j(\rho), t\geqslant 0\}$. It follows that there exists an increasing to infinity sequence $\{t_n\}$ such that for every $\rho\in Tr(\mathscr{H}_q)$

$$P(\rho)=w^*\text{-}\lim_{n\to\infty} j\left(\frac{1}{t_n}\int_0^{t_n} ds\, T_{s*}\rho\right) \tag{3}$$

Remark. On the subspace $Tr(\mathscr{H}_q)_0=\{\rho\in Tr(\mathscr{H}_q) : Tr\rho=0\}$, which is equal to the norm closure of Range B_* in $Tr(\mathscr{H}_q)$ we can define a T_*-invariant projector P_0: $Tr(\mathscr{H}_q)_0\to Tr(\mathscr{H}_q)$ (see Theorem 5.1 in ref. 10). Between P_0 and P there is the following relation.[11] $P(\rho)\in Tr(\mathscr{H}_q)$ if and only if $\rho\in Tr(\mathscr{H}_q)_0$ and for such ρ there is $P(\rho)=P_0(\rho)=0$, the second equality holds because P_0 is the projector onto Fix(T_*) in $Tr(\mathscr{H}_q)$.

Proposition 3.12. $P(\rho)=(Tr\rho)\,\omega$ for some $\omega\in\text{Fix}(T^*)$. Hence dim Range $P=1$.

Proof. If $Tr(\rho)=0$ then $P(\rho)=0$ by the above remark. We show that for any ρ_1 and ρ_2 such that $Tr\rho_1=Tr\rho_2=1$ there is $P(\rho_1)=P(\rho_2)$. Let $A\in B(\mathscr{H}_q)$. Then

$$|\langle A, P(\rho_1-\rho_2)\rangle| = \lim_{n\to\infty} \left|\left\langle A, j\left(\frac{1}{t_n}\int_0^{t_n} ds\, T_{s*}(\rho_1-\rho_2)\right)\right\rangle\right|$$
$$= \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} ds\, |\langle T_{s*}(\rho_1-\rho_2), A\rangle|$$
$$\leqslant \|A\|_{op} \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} ds\, \|T_{s*}(\rho_1-\rho_2)\|_{Tr} = 0$$

since $\|T_{s*}(\rho_1-\rho_2)\|_{Tr} \to 0$ for $s\to\infty$. So $P(\rho_1-\rho_2)=0$. ▮

Corollary. For the same sequence $\{t_n\}$ we have that

$$w^*\text{-}\lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} ds\, T_s A = \omega(A)\,\mathbf{1}$$

for every $A \in B(\mathcal{H}_q)$.

Proof. It follows from formula (3) and Proposition 3.12. ▮

Finally, let us describe the space $\mathrm{Fix}(T^*)$. To do this we introduce the ultrapower $\hat{B}(\mathcal{H}_q)$ of the von Neumann algebra $B(\mathcal{H}_q)$. Let $\mathcal{U}$ be a free ultrafilter (i.e., not generated by a single set) on the set of natural numbers $\mathbf{N}$. If $l^\infty(B(\mathcal{H}_q))$ is the Banach space of all uniformly bounded functions $\mathbf{N} \to B(\mathcal{H}_q)$, then

$$c_{\mathcal{U}}(B(\mathcal{H}_q)) = \{(A_n) \in l^\infty(B(\mathcal{H}_q)) : \lim_{\mathcal{U}} \|A_n\|_{op} = 0\}$$

is a Banach subspace. For the definitions of an ultrafilter and the limit of a filter see for example ref. 21. By the ultrapower we understand the quotient Banach space

$$\hat{B}(\mathcal{H}_q) = l^\infty(B(\mathcal{H}_q))/c_{\mathcal{U}}(B(\mathcal{H}_q))$$

Because $\mathcal{U}$ is an ultrafilter, so the norm of an abstract class $[(A_n)] \in \hat{B}(\mathcal{H}_q)$ is given by $\|[(A_n)]\| = \lim_{\mathcal{U}} \|A_n\|_{op}$.[32] It is clear that $c_{\mathcal{U}}(B(\mathcal{H}_q))$ is a two sided ideal in $l^\infty(B(\mathcal{H}_q))$ and so $\hat{B}(\mathcal{H}_q)$ is a C^*-algebra. There is a canonical embedding of $B(\mathcal{H}_q)$ into $\hat{B}(\mathcal{H}_q)$, $A \to [(A, A,...)]$, which is an isometry. Moreover, every $T \in \mathcal{L}(B(\mathcal{H}_q))$ has an extension (also called canonical) to an operator $\hat{T}$ on $\hat{B}(\mathcal{H}_q)$ given by $\hat{T}[(A_n)] = [(TA_n)]$.

Theorem 3.13. $\dim \mathrm{Fix}(T^*) = \infty$.

Proof. Suppose that the contrary is true, i.e., $\dim \operatorname{Fix}(T^*) < \infty$. Because T_t^* is a contraction, so

$$\operatorname{Fix}(T^*) = \operatorname{Fix}(\lambda R(\lambda, B^*)) \equiv \operatorname{Fix}(R^*)$$

By Theorem 4.4 Section D-IV in ref. 16 we obtain that $\dim \operatorname{Fix}(\hat{R}) < \infty$, where $\hat{R}(\lambda, B)$ is the canonical extension of the resolvent $R(\lambda, B)$, $\lambda > 0$, onto some ultrapower $\hat{B}(\mathscr{H}_q)$. Hence, by Proposition 2.3 Section D-III in ref. 16, point 0 is a pole of the resolvent $R(\lambda, B)$ such that the corresponding residue has finite rank. Using the results of Groh[17, 18] we conclude that every T^*-invariant state is normal. Hence we get a contradiction with Proposition 3.11. ▮

Example. Let us consider a classical particle moving freely, i.e., along a geodesic curve, on the Lobatchevski space

$$Q = \mathbf{R} \times \mathbf{R}_+ = \{(q_1, q_2) : q_2 > 0\}$$

Let us recall that a geodesic curve is a vertical straight line or a semicircle with the center placed in an arbitrary point on the x_1-axis. This particle interacts with a quantum particle on the same space. To describe the quantum system we use the system of generalized coherent states on Q.[29] Let us recall that Q is a homogeneous space $Q = SL(2, \mathbf{R})/SO(2)$. For simplicity we take the first representation from the series $(\mathscr{H}_k, \pi_k)$, where $k = 1, 3/2, 2, \ldots$. That is

$$\mathscr{H}_q = \left\{ f : \|f\|^2 = \int d\mu_1(z)\, |f(z)|^2 < \infty \right\}$$

where f is a holomorphic function in the unit complex disc $|z| < 1$ and $d\mu_1 = (1/\pi)\, dz\, d\bar{z}$. For $q = (q_1, q_2) \in Q$ we have one-dimensional projectors $P_q = |\zeta\rangle\langle\zeta|$, where

$$|\zeta\rangle = \frac{1 - |\zeta|^2}{(1 - \bar{\zeta} z)^2} \qquad \text{and} \qquad \zeta = \frac{1 - q_2 + i q_1}{1 + q_2 - i q_1}$$

The quantum operators are given by

$$\hat{f} = \int_Q d\alpha(q)\, f(q)\, P_q$$

where $d\alpha$ is the unique $SL(2, \mathbf{R})$ invariant measure on Q normalized in such a way that $\int_Q P_q\, d\alpha(q) = \mathbf{1}$, the identity operator. The modification of the classical paths through the interaction was discussed in ref. 6. Because in this case the reproducing kernel is a holomorphic function such that $|K(q, q')|^2 = TrP_q P_{q'} > 0$, all results concerning the asymptotic behavior of the quantum system hold. Because $T_t: K(\mathscr{H}_q) \to K(\mathscr{H}_q)$, so we can define the quotient semigroup $\tilde{T}_t$ on the Calkin algebra $\mathscr{C} = B(\mathscr{H}_q)/K(\mathscr{H}_q)$, $\tilde{T}_t[A] = [T_t A]$. Because any state $\omega \in \mathrm{Fix}(T^*)$ is singular, so it induces a state $\tilde{\omega}$ on $\mathscr{C}$, $\tilde{\omega}[A] = \omega(A)$. Clearly $\tilde{\omega} \in \mathrm{Fix}(\tilde{T}^*)$. On this example we show that even $\tilde{\omega}$ may not be faithful.

Proposition 3.14. There exists an infinite dimensional projector Q such that $\omega(Q) = 0$.

Proof. Because $T_t^*(\omega) = \omega$ for all $t \geqslant 0$, so it is enough to show that

$$\lim_{t \to \infty} \|T_t Q\|_{op} = \lim_{t \to \infty} \|S_t Q\|_{op} = 0$$

Let $B_0(\mathscr{H}_q)$ denote the linear subspace in $B(\mathscr{H}_q)$ on which the semigroup S_t is stable, i.e., $\lim_{t \to \infty} \|S_t A\|_{op} = 0$ iff $A \in B_0(\mathscr{H}_q)$. It is clear that $B_0(\mathscr{H}_q)$ is closed in the operator norm. By Corollary 3.4 in ref. 31 we have that

$$\bigcup_{s > 0} \mathrm{Range}(S_s - id) \subset B_0(\mathscr{H}_q)$$

Hence also $\mathrm{Range}(L) \subset B_0(\mathscr{H}_q)$. Let $|n\rangle = \sqrt{n+1}\, z^n$, $n \in \mathbf{N} \cup \{0\}$. Then $\{|n\rangle\}$ form an orthonormal base in $\mathscr{H}_q$.[29] It was shown in ref. 28 that for $P_n = |n\rangle\langle n|$ the following formula holds

$$L(P_n) = \sum_{m=0}^{\infty} \Pi(n, m)\, P_m - P_n$$

where $\Pi(n, m)$ is a symmetric probability kernel on $\mathbf{N} \cup \{0\}$ given by

$$\Pi(n, m) = \frac{2(m+1)(n+1)}{(n+m+1)(n+m+2)(n+m+3)}$$

It means that $\Pi(n, m) = \Pi(m, n)$ and $\sum_m \Pi(n, m) = 1$. Let us choose a subsequence of natural numbers $\{n_k\}$ (for example $n_k = 2^k$) such that $b_m = \sum_k \Pi(n_k, m)$ satisfies $\lim_{m \to \infty} b_m = 0$. Let $\mathscr{H}_0$ be a closed subspace generated by $\{|n_k\rangle\}$, $k \in \mathbf{N}$ and let Q denote the projector onto $\mathscr{H}_0$. Because

$$P_q Q P_q = Tr(QP_q)\, P_q = \left[\sum_{k=1}^{\infty} Tr(P_{n_k} P_q)\right] P_q$$

so

$$\int_Q d\alpha(q)\, P_q Q P_q = \sum_{k=1}^{\infty} \int_Q d\alpha(q)\, Tr(P_{n_k} P_q)\, P_q = \sum_{k=1}^{\infty} \left[\sum_{m=0}^{\infty} \Pi(n_k, m)\, P_m \right]$$

$$= \sum_{m=0}^{\infty} b_m P_m \in K(\mathscr{H}_q) \subset B_0(\mathscr{H}_q)$$

Because $L(Q) \in B_0(\mathscr{H}_q)$, so also $Q \in B_0(\mathscr{H}_q)$. ∎

4. CONCLUDING REMARKS

In the present paper we continue the analysis of the properties of a possible classical-quantum coupling when the interaction between the classical and quantum system is given by a completely positive semigroup. Although the semigroup describes the time evolution of the total system we can also consider the behavior of its classical and quantum part. In the previous paper[6] we concentrated on the construction of the associated piecewise deterministic process and on the modification of classical paths through the interaction. Here we considered the asymptotic behavior of the quantum subsystem. It was shown that for infinite dimensional quantum systems it is impossible to establish the limit $\lim_{t \to \infty} Tr\rho(t)\, A$ for all observables A. In order to obtain such a strong ergodic property one has to restrict the set of observables to a suitable T-invariant subspace $\mathscr{A} \subset B(\mathscr{H}_q)$. For example, we can define $\mathscr{A}$ as the set of all A such that $\omega_1(A) = \omega_2(A)$ for any $\omega_1, \omega_2 \in \mathrm{Fix}(T^*)$. Then, for such A, there would be exactly one invariant state and we could use the qualitative stability theorem. The role of an actualized subset of observables was discussed in ref. 30. In general, the weak*-closure of the convex hull of the set $\{ j(T_{t*} \rho), t \geqslant 0\}$ contains infinitely many invariant singular states. Because $\mathrm{Fix}(T_*) = 0$ does not separate $\mathrm{Fix}(T) = \mathbf{C}\mathbf{1}$, and $\mathrm{Fix}(T)$ does not separate $\mathrm{Fix}(T^*)$, we can not expect a more precise description of the ergodic properties of the semigroup T_t. This differs significantly from the case of finite dimensional quantum systems. It is also different from the asymptotic behavior of the coupled spin-boson system. Such a coupling of a spin $\frac{1}{2}$ system with an infinitely extended free Bose gas at positive temperature was discussed in ref. 19. It was shown there that for a coupling of Hamiltonian type with a small coupling parameter λ the total system has the property of return to equilibrium. In the limit $\lambda \to 0$ the partial trace of a total density matrix with respect to the reservoir variables gives, at $t \to \infty$, a Gibbs state on the matrix algebra $M_{2\times 2}(\mathbf{C})$.

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